Fall 2013 **Math 601** Name: ______________________________

Quiz 8

Question 1. (8 pts)

We have learned that if two square matrices A and B are similar, then A and B have the same eigenvalues. In this question, let us see what is the relation between the eigenvectors of A and the eigenvectors of B.

(a) Accept as a fact that $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ is similar to the diagonal matrix $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$. In fact, we have

$$D = P^{-1}AP$$

where $P = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$. Find the eigenvectors of A and D for the eigenvalue $\lambda = 5$. Are they the same up to a scalar constant? In other words, are they multiple of each other?

Solution: The eigenvector of A belonging to 5 is

$$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

which is the first column of P. The eigenvector of D belonging to 5 is

$$w = e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Clearly, they are not multiple of each other.

(b) In general, suppose B is similar to C, say $B = Q^{-1}CQ$. If v is an eigenvector of B belonging to eigenvalue λ_0, then show that Qv is an eigenvector of C belonging to the same eigenvalue λ_0.

Solution: Since $B = Q^{-1}CQ$, we have $C = QBQ^{-1}$. Therefore,

$$C(Qv) = QBQ^{-1}(Qv) = QB(v) = Q(\lambda_0 v) = \lambda_0 Qv.$$

So Qv is an eigenvector of C belonging to the eigenvalue λ_0.

Question 2. (12 pts)

Find the general solution to the following system of differential equations:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 4 & -3 \\ -2 & 2 & -1 \end{bmatrix}$.

Solution:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} -\lambda & 2 & -3 \\ -2 & 4-\lambda & -3 \\ -2 & 2 & -1-\lambda \end{vmatrix} \\
&\text{cofactor expansion along the first row} \\
&= -\lambda(\lambda^2 - 3\lambda + 2) - 2(2\lambda - 4) - 3(4 - 2\lambda) \\
&= -\lambda(\lambda - 1)(\lambda - 2) - 4(\lambda - 2) + 6(\lambda - 2) \\
&= [-\lambda(\lambda - 1) + 2](\lambda - 2) \\
&= -(\lambda^2 - \lambda - 2)(\lambda - 2) = -(\lambda + 1)(\lambda - 2)^2
\end{aligned}$$

So the eigenvalues are -1 and 2. When $\lambda = -1$, then

$$v_1 = (1, 1, 1)^T.$$

When $\lambda = 2$, there are two linearly independent eigenvectors

$$v_2 = (1, 1, 0)^T, v_3 = (-3, 0, 2)^T.$$

So the general solution is

$$\mathbf{x}(t) = c_1 e^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

In other words,

$$\begin{cases} x_1(t) = c_1 e^{-t} + c_2 e^{2t} - 3c_3 e^{2t} \\ x_2(t) = c_1 e^{-t} + c_2 e^{2t} \\ x_3(t) = c_1 e^{-t} + 2c_3 e^{2t} \end{cases}$$